551.574.12 : 551.576.11

Cloud-droplet growth by condensation in cumulus

By B. J. MASON and C. W. CHIEN*

Department of Cloud Physics, Imperial College, London

University of California, Los Angeles

(Manuscript received 25 July 1961; in revised form 27 November 1961)

SUMMARY

The growth of droplets by condensation upon a spectrum of salt nuclei is followed as they are carried up in a small cumulus which mixes and exchanges heat, momentum and water vapour with its surroundings. The vertical velocity, water content, supersaturation and cumulative drop-size distributions are plotted as functions of time and height above cloud base. Growing cloud droplets are continually being transferred from the cloud to the drier environment and replaced by new condensation nuclei. The fact that some droplets remain in the cloud longer than others is largely responsible for the evolution of a broad drop-size distribution which much more closely resembles observed spectra than those obtained from previous calculations for a closed parcel. Regarding the formation of large droplets, the calculations show that droplets exceeding 20 μ in radius arise on salt nuclei of mass 10^{-10} g and droplets larger than 30 μ radius arise on nuclei of 10^{-9} g within ten minutes of the air passing through the condensation level. Continued growth in a succeeding thermal and further deformation of the droplet spectrum by coalescence will produce even larger drops, but nevertheless it is difficult to explain the appearance of droplets larger than 30 μ in concentrations of order 1 cm^{-3} in terms of their growth on reported concentrations of giant hygroscopic nuclei.

1. INTRODUCTION

A number of attempts have been made to compute the evolution of cloud-droplet spectra, notably by Howell (1949), Squires (1952), Neiburger and Chien (1959), and Mordy (1959), all of whom considered a closed parcel of air containing a specified population of condensation nuclei to rise at an arbitrarily assigned vertical velocity and to cool uniformly and, in most cases, adiabatically, throughout its volume. Such a parcel does not mix with its surroundings and so its contents are conserved. All these computations agree in showing that, with the closed parcel, the peak supersaturation is reached within a few tens of seconds (corresponding usually to an ascent of a few tens of metres) of the air becoming saturated and that, thereafter, the droplet spectrum narrows rather rapidly. Even if the initial condensation nuclei show a wide dispersion, the resulting droplet-size spectra are much narrower than are found in natural clouds, and the model cannot be made to produce realistic spectra by varying the updraught speed or its variation with height. Only the few very large nuclei, which are unable to attain the equilibrium size appropriate to the prevailing supersaturation, are able to retain much of their initial size differential. It is highly unlikely that the concentration and size distribution of the droplets are, in fact, largely determined during the first few tens of metres of ascent above cloud base and that, during the remainder of the ascent, the droplets grow ever more slowly with the smaller ones catching up on the larger to narrow the spectrum.

The most unrealistic features of the closed parcel model are :

(1) the assumption that the cloudy air does not mix and become diluted with the surrounding drier air; the reality of mixing is indicated by the measured liquid-water content of non-precipitating clouds being usually less than the adiabatic value; (2) the assumption that *all* droplets are retained and have equal life-times in the cloud; (3) the assignment of an arbitrary vertical velocity since, as Mason and Emig (1961) have shown, a closed saturated parcel would undergo a strongly oscillatory vertical motion.

* Now at A. and M. College of Texas, College Station, Texas.

Mason and Emig (1961) computed how the vertical velocity, temperature, humidity-mixing ratio and liquid-water content of a saturated parcel vary with height and time as it ascends from the condensation level, rises through, and mixes with, its surroundings. This simplified treatment, which assumes the thermal to possess horizontal homogeneity, arrives at rather critical values for a single parameter used to represent the rate of mixing. It will be used in this paper as the dynamical framework for a computation of the growth of a population of cloud droplets in which, in addition to exchanging heat, momentum and water vapour, the mixing process is assumed to transfer droplets from the cloud to the drier environmental air where they evaporate and are replaced by fresh condensation nuclei of the same mass. It has been shown by Mason (1952, 1960) that the turbulent transfer of droplets to the cloud boundaries, which ensures that droplets have non-equal life-times in the cloud, can largely account for the observed breadth of droplet-size spectra in *stratus* clouds.

2. Droplet growth by condensation

To formulate the problem we need differential equations for the vertical motion of the cloudy air, for the rates-of-change of temperature, supersaturation, liquid-water content, and for droplet growth around condensation nuclei of specified mass. It is also necessary to specify the temperature, pressure and vertical velocity of the air and the *initial* sizes of the droplets at the condensation level; also the temperature and humidity distribution in the environment, and a value for the mixing parameter.

The equation for the vertical motion of the parcel is :

$$\frac{dU}{dt} = \left(\frac{T - T'}{T} - w\right) g - \left(k + \frac{1}{M}\frac{dM}{dt}\right) U \ , \qquad . \qquad . \qquad (1)$$

where U is the vertical velocity, T, T' the temperature of the parcel and the environment, M the mass of the parcel, w its liquid-water content in grammes per gramme of air, and k is Priestley's exchange coefficient. We shall follow Mason and Emig (1961) in one of their computations and write

$$\left(k + \frac{1}{M}\frac{dM}{dt}\right) = k' = \text{const.}$$

and assume that this damping factor is the same for the transfer of heat, momentum and water.

We can no longer retain the simplification that the air remains just saturated because it is the slight degree of supersaturation which controls the rate of droplet growth. The following accurate equations are therefore rather more complicated than those by Mason and Emig. The rate-of-change of temperature may now be written :

$$\frac{T}{t} = -\frac{\dfrac{gU}{c_p}\left[1 + \dfrac{L}{R_d T}\left(\dfrac{\epsilon e_s}{p - e_s}\right)\right] - k'(T - T') - \dfrac{k'L}{c_p}\left[(1+\sigma)\dfrac{\epsilon e_s}{p - e_s} - \dfrac{\epsilon e_s' H'}{p - e_s'}\right] - k'\dfrac{c_w}{c_p}(wT - w'T')}{\left[1 + \dfrac{\epsilon L^2}{R_d c_p T^2}\left(\dfrac{\epsilon e_s}{p - e_s}\right)\right]} \qquad (2)$$

where p, e_s, σ are respectively, the total pressure, saturation vapour pressure and supersaturation of the cloudy air; H', e_s' the saturation ratio and saturation vapour pressure of the environment; L the latent heat of condensation; R_d the gas constant for dry air; c_p, c_w the specific heats of dry air and of liquid water, and ϵ the ratio of the densities of water vapour and dry air.

The rates-of-change of liquid-water content w and humidity-mixing ratio x are related as follows :

$$\frac{dw}{dt} = -\frac{dx}{dt} - k'(x - x' + w - w') \quad . \quad . \quad . \quad (3)$$

The supersaturation of the vapour $\sigma = (x/x_s) - 1$,

so $$\frac{d}{dt}(1+\sigma)x_s = (1+\sigma)\frac{dx_s}{dt} + x_s\frac{d\sigma}{dt} = -\frac{dw}{dt} - k'(x - x' + w - w'),$$

or $$\frac{d\sigma}{dt} = -\frac{1}{x_s}\left[\frac{dw}{dt} + k'(x - x' + w - w')\right] - (1+\sigma)\frac{1}{x_s}\frac{dx_s}{dt}.$$

Since $$x_s = \frac{\epsilon e_s}{p - e_s}, \quad \frac{1}{e_s}\frac{de_s}{dt} = \frac{\epsilon L}{R_d T^2}\frac{dT}{dt}, \quad \text{and} \quad \frac{1}{p}\frac{dp}{dt} = -\frac{gU}{R_d T},$$

$$\frac{1}{x_s}\frac{dx_s}{dt} = \left(\frac{\epsilon L}{R_d T^2}\frac{dT}{dt} + \frac{gU}{R_d T}\right).$$

The conditions in the environment are represented by

$$\frac{dT'}{dt} = U\frac{dT'}{dz}, \quad x' = \frac{H'\epsilon e_s'}{p - e_s'} \text{ and so, finally,}$$

$$\frac{d\sigma}{dt} = -\frac{(p - e_s)}{\epsilon e_s}\left\{\frac{dw}{dt} + k'(w - w') + k'\left[(1+\sigma)\frac{\epsilon e_s}{p - e_s} - \frac{H'\epsilon e_s'}{p - e_s'}\right]\right\}$$

$$- (1+\sigma)\left[\frac{\epsilon L}{R_d T^2}\frac{dT}{dt} + \frac{gU}{R_d T}\right] \quad . \quad . \quad . \quad (4)$$

The growth rate of a droplet forming upon a salt nucleus of mass m and molecular weight W may be calculated from the following equation derived by Mason and Ghosh (1957) :

$$r\frac{dr}{dt} = \left[\sigma - \frac{2\gamma M_w}{\rho RTr} + \frac{8{\cdot}6\,\mathrm{m}}{Wr^3}\right] \Big/ \left[\frac{L^2 M_w \rho}{KRT^2} + \frac{\rho RT}{DM_w e_s}\right] \quad . \quad . \quad . \quad (5)$$

where γ, M_w and ρ are respectively the surface tension, molecular weight and density of water, K the thermal conductivity of air, D the diffusion coefficient of water vapour in air, and R the universal gas constant. We have, in addition,

$$\frac{dw}{dt} = \frac{4\pi}{3}\left[\frac{d}{dt}\sum_r n_r r^3 - k'\left(\sum_r n_r r^3 - \sum_{r'} n_{r'} r'^3\right)\right] , \quad . \quad . \quad . \quad (6)$$

where n_r is the number of droplets per gramme of air of radius r in the cloud and n' is the corresponding concentration of nucleus-droplets of radius r' in the environment.

Starting from the initial conditions listed in Table 1, Eqs. (1), (2), (4), (5) and (6) were integrated numerically in sufficiently small time-steps to ensure computational stability on the IBM 709 high-speed computer, to give values of U, T, $(T - T')$, σ, w, r for each nucleus group, p, and the height of the parcel z, which were printed out every 15 sec.

The initial values of the parameters were chosen to give, according to the computations of Mason and Emig (1961), a small cumulus only about 1,200 m (4,000 ft) deep, with a maximum updraught of 3 m sec^{-1}, and a maximum liquid-water content of about 0·8 g kg^{-1}. Such a cloud is fairly representative of small non-precipitating cumulus in which drop sizes have been measured and are available for comparison. The condensation-nucleus spectrum obeys a simple inverse mass law, the initial sizes of the droplets formed around the smallest nuclei corresponding to their equilibrium radii in saturated air, while the largest nucleus droplets were assumed to be droplets of sea water.

TABLE 1. INITIAL CONDITIONS OF THE COMPUTATION

Temperature and pressure at condensation level, 10°C, 900 mb
Initial velocity of parcel $U_0 = 50$ cm sec^{-1}
Initial excess temperature $T - T' = 0{\cdot}7$°C
Lapse rate of environment $= 7$°C km^{-1}
Humidity of environment H' $= 90$ per cent
Mixing parameter $k' = 4 \times 10^{-3}$ sec^{-1}

Mass of nucleus (g of NaCl)	10^{-15}	10^{-14}	10^{-13}	10^{-12}	10^{-11}	10^{-10}
Concentration of nuclei (g^{-1})	3×10^5	3×10^4	3×10^3	3×10^2	30	3
Initial droplet radius (μ)	0·34	1·0	2·2	3·7	6·0	10·0

The concentrations and sizes of the nuclei entrained through the sides of the cloud were taken as being the same as for those entering the cloud base. Since Twomey (1955) and Byers, Sievers and Tufts (1957) found that, in the absence of precipitation, inversions and large cumulus, chloride nuclei overland were mixed fairly uniformly up to heights of a few kilometres, there seems little reason for assuming the nucleus population to fall appreciably with increasing height in the clear air surrounding our cloud. Although it might be more reasonable to assume the nuclei surrounding the cloud to be in equilibrium with the environmental humidity, no appreciable error is introduced in assuming that they enter the cloud with the initial sizes given in Table 1 because even the largest nuclei would grow to those sizes in just saturated air within 20 sec.

The magnitude of the computation was greatly enlarged by the fact that droplets are continually being transferred to the cloud edges where they evaporate and are replaced by new inactivated nuclei so that, in each nucleus-size group, the number of size classes was increased after every time step. To keep track of all the droplet-size classes a programme was written which enabled the machine to compute, at each time-step, the cumulative rather than the differentiated drop-size distribution for each of the six nucleus-size groups. The computed droplet-size spectra are therefore plotted in cumulative form. The whole computation occupied nearly the whole memory of the 709 machine and took nearly 6 hours of machine time.

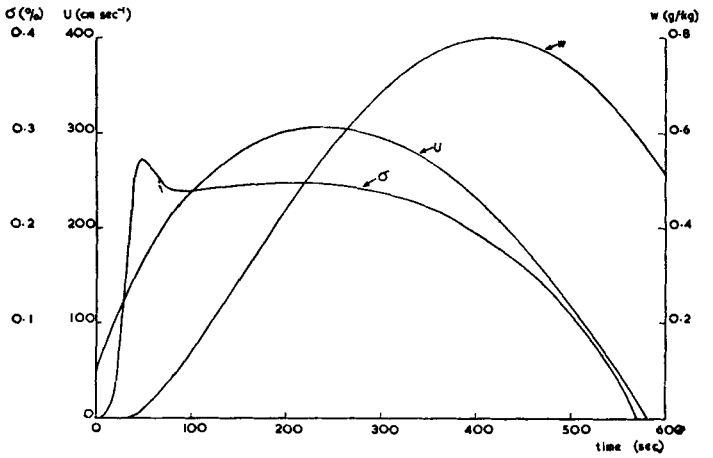

Figure 1. The variation with time of the vertical velocity U, liquid-water content w and supersaturation σ.

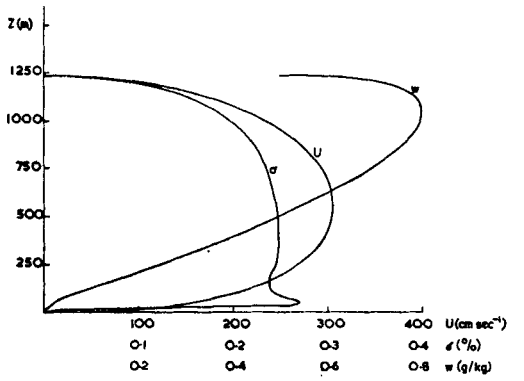

Figure 2. The variation with height above cloud base of vertical velocity, U, liquid-water content w, and supersaturation σ.

The variation with time of vertical velocity U, liquid-water content w and supersaturation σ are plotted in Fig. 1. The vertical velocity of the parcel reaches a maximum value of 306 cm sec^{-1} (240 sec and 556 m after it passes the condensation level) and reaches zero again after 580 sec at a height of 1,233 m. The liquid-water content reaches its maximum value of 0·8 g/kg (about four-tenths of the adiabatic value) after 420 sec at 1050 m above cloud base.

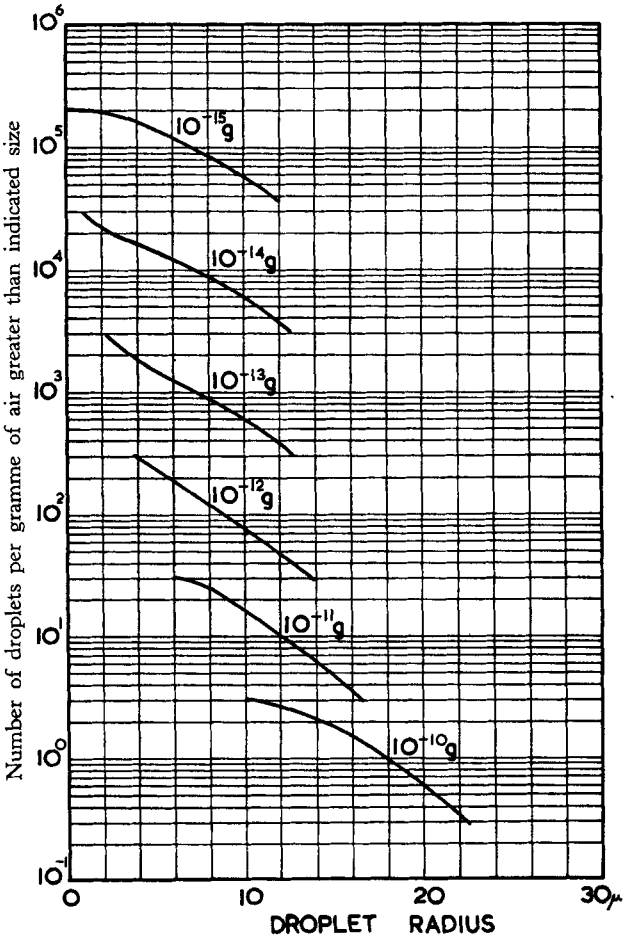

Figure 3. The cumulative drop-size distributions for each nucleus-size group reached after 580 sec.

The supersaturation rises steeply at first and attains a peak value after only 45 sec (48 m) because, during this period, there is practically no growth of the two most numerous nucleus groups – a reflection of the fact that the values chosen for their initial radii were slightly too large. But this rather artificial situation is soon adjusted and, after 90 sec, the supersaturation rises slowly to another peak value of 0·25 per cent after 220 sec (495 m), and thereafter continues to fall. The variation with height of U, σ and w is plotted in Fig. 2.

The cumulative drop-size distributions attained at the instant when the updraught ceases are plotted for each nucleus-size group in Fig. 3. During the growth period of 580 sec, the droplet spectrum, which initially extended from 0·34 μ to 10 μ radius, has continually spread and, finally, extends from 0·34 μ to 22·5 μ. But those droplets which have remained in the cloud for the whole 580 sec range only from 12·1 μ to 22·5 μ. The variations of median and volume-mean radii with height are shown in Fig. 4. After nearly 10 min droplets growing on salt nuclei of mass 10^{-10} g attain a radius of 22·5 μ while nuclei of 10^{-9} g would, under these conditions, attain a radius of 33 μ.

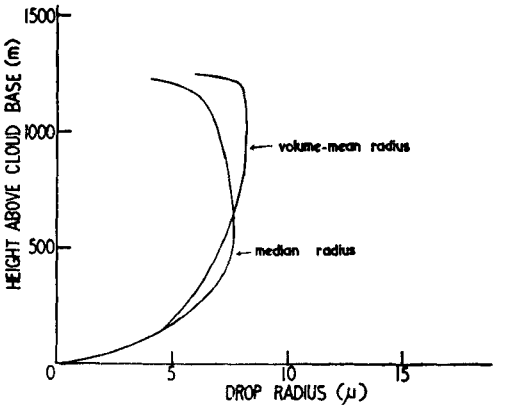

Figure 4. The median and volume-mean radii of the cloud droplets as a function of height in the cloud.

The growth of these drops would be prolonged still further if they were to survive, enter and continue their growth in a newly-rising thermal. If the average supersaturation during the life of the new thermal were only 0·2 per cent, drops growing by condensation and coalescence about nuclei of 10^{-11} g would attain a radius of 30 μ, and those growing on nuclei of 10^{-9} g would reach 50 μ in a total time of 30 min. Once droplets of this size are formed, the spectrum becomes deformed rather rapidly by coalescence; but since Hocking's (1959) theoretical values of collision efficiency and Picknett's (1960) confirmatory experimental measurements show that droplets grow to 30 μ radius predominantly by condensation, we may compare the results of the above computations with those of previous theoretical treatments and with observed droplet spectra in small cumulus in so far as we are concerned with the production of droplets of up to 30 μ radius.

3. Comparison of computed and observed droplet spectra

It is instructive to compare the results of the above calculations with the most similar of a recent series of computations by Mordy (1959) which treated a closed parcel rising at a constant upward velocity of 1 m sec^{-1} and containing a total of about 450 salt nuclei per cm^3 ranging from 10^{-17} g to $10^{-8\cdot5}$ g. In Mordy's example a peak supersaturation of 1 per cent is reached after only 40 sec (or 40 m above cloud base) at which stage, droplets growing on nuclei of 10^{-15} g have increased in radius by 2 μ and those on nuclei of 10^{-10} g by only 1 μ. Thereafter the spectrum begins to narrow as the supersaturation falls and the largest nuclei grow only very slowly. Thus the main character of the spectrum is determined during the first minute of cloud growth and little development occurs during the rest of its life.

The results of our treatment appear much more realistic in that the peak supersaturation of 0·25 per cent is reached after 220 sec and nearly half-way up the cloud; droplets growing on those nuclei of 10^{-15} g which have remained in the cloud during this period have increased in radius by 8 μ, while those orginating on nuclei of 10^{-10} g have grown by 8·7 μ. Thereafter the spectrum continues to broaden until, after 10 min, it ranges from 0·34 μ to 22·5 μ, and extends to 33 μ if nuclei of mass 10^{-9} g are present. Moreover, in our cloud having a depth of 1234 m, the magnitude and variation with height of the liquid-water content and mean-volume droplet radius (see Figs. 2 and 4) are rather similar to those reported by Durbin (1956) on the basis of aircraft measurements in *non-precipitating* cumulus, 4,000 to 5,000 ft in depth, over southern England.

There is, however, some difficulty in explaining Durbin's observations that droplets of radius greater than 30 μ sometimes occur in concentrations as high as 1 cm^{-3}. According to our calculations, such droplets could arise within 10 min on nuclei of mass 10^{-9} g, and upon nuclei of 10^{-11} g if they continued their growth in a succeeding thermal over a total period of about 30 min. However, the observed concentrations of hygroscopic nuclei of $m = 10^{-11}$ g rarely exceed 100/litre and more often are of order 10/litre at cloud level. Since there are, unfortunately, no simultaneous measurements of large hygroscopic nuclei and of drop sizes within a cloud, there must be some uncertainty in a discussion on whether the large drops may be accounted for by condensation on the large nuclei, but the data which we have do suggest that droplets of $r > 30$ μ are sometimes at least an order of magnitude more numerous than the hygroscopic nuclei on which they might grow within a reasonable time. However, one must be very cautious in correlating two sets of data taken at different places and under different circumstances, and much more careful observations, not only of droplet and nucleus populations, but also of cloud structure and evolution will be required to determine to what extent large hygroscopic nuclei are responsible for extending the drop-size spectrum to sizes at which the coalescence process may take over and continue development towards the precipitation stage.

Acknowledgments

This work was carried out whilst one of us (B.J.M.) was on leave as Visiting Professor of Meteorology at the University of California, Los Angeles, and was partly sponsored under the National Science Foundation Contract 6-5095.

The authors wish to acknowledge gratefully the use, without charge, of the IBM 709 computer and the other facilities of the Western Data Processing Centre, University of California, Los Angles.

References

Byers, H. R., Sievers, J. R. and Tufts, B. J. 1957 *The artificial stimulation of rain*, Pergamon Press, New York, p. 47.

Durbin, W. G. 1956 *Air Ministry, Met. Res. Cttee., M.R.P.*, No. 991.

Hocking, L. M. 1959 *Quart. J. R. Met. Soc.*, **85**, p. 44.

Howell, W. E. 1949 *J. Met.*, **6**, p. 134.

Mason, B. J. 1952 *Quart. J. R. Met. Soc.*, **76**, 377.

1957 *The physics of clouds*, Clarendon Press, Oxford, p. 424.

1960 *J. Met.*, **17**, p. 459.

Mason, B. J. and Emig, R. 1961 *Quart. J. R. Met. Soc.*, **87**, p. 212.

Mason, B. J. and Ghosh, D. K. 1957 *Ibid.*, **83**, p. 501.

Mordy, W. A. 1959 *Tellus* **11**, p. 16.

Neiburger, M. and Chien, C. W. 1959 'Physics of precipitation,' *Geophys. Res. Monogr. No. 5.* (Am. Geophys. Un.), p. 191.

Picknett, R. G. 1960 *Aerodynamic capture of particles*, Pergamon Press, London, p. 160.

Squires, P. 1952 *Austr. J. Sci. Res.*, **A5**, 1, p. 473.

Twomey, S. 1955 *J. Met.*, **12**, p. 81.